The Primacy of the Universal Quantifier in Frege's Concept-Script

Joongol Kim

The Primacy of the Universal Quantifier in Frege's Concept-Script

Joongol Kim

This paper presents three explanations of why Frege took the universal, rather than the existential, quantifier as primitive in his formalization of logic. The first two explanations provide technical reasons related to how Frege formalizes the logic of truth-functions and the logic of quantification. The third, philosophical explanation locates the reason in Frege's logicist goal of analyzing arithmetical concepts—especially the concepts of 0 and 1—in purely logical terms.

It is a well-known fact of elementary logic that each of the universal and existential quantifier symbols, ∀ and ∃, can be defined in terms of the other, as follows:

(1) $\exists_\alpha \phi =_{df} \neg\forall_\alpha \neg\phi$
(2) $\forall_\alpha \phi =_{df} \neg\exists_\alpha \neg\phi$.

So one could adopt ∃ as a primitive symbol and then define ∀ in terms of it. Frege, the inventor of modern quantificational logic, did the reverse, taking the universal quantifier as primitive in his formalization of logic called the *concept-script*.[1] Thus, in his early monograph on the concept-script, *Begriffsschrift*, Frege (1967, sec.11) introduces his universal quantifier symbol—the concavity, ⌣ —and expresses "$\forall_\alpha \phi$" as follows:

$$\vdash\!\!\overset{\mathfrak{a}}{\smile}\; \Phi(\mathfrak{a})$$

Then, using the negation stroke, ⊤ , Frege (1967, sec.12) constructs the complex formula

$$\vdash\!\!\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\; \varLambda(\mathfrak{a})$$

---

1 For a quick introduction to Frege's concept-script, see Cook (2013).

and reads it as "There are Λ," although, taken literally, it says that not all things are non-Λ. Frege's concept-script includes no special existential quantifier symbol such as the downside-up form of the concavity—the *convexity* (see Kneale and Kneale 1962, 516–17)—as an abbreviation of $\overline{}\smile\overline{}$.

An interesting question is why Frege employed the universal, rather than the existential, quantifier as a primitive sign in his formal language. Nowhere in his writings does he address this question. Indeed, as Macbeth (2005, 4) noted, "it seems never even to occur to him that he could treat the existential quantifier as the primitive sign for generality and then define the universal quantifier in terms of it." The main purpose of this paper is to address this gap in our understanding of Frege's logical formalism by giving three possible explanations of Frege's adoption of the universal quantifier as a primitive.

The first two explanations—to be discussed in turn in sections 1, 2—offer technical reasons: given how the logic of truth-functions and the logic of quantification are formalized in the concept-script, it was natural and convenient to take the universal quantifier as primitive. The third explanation—to be given in section 3—is that Frege was forced to adopt the universal quantifier as a primitive in his pursuit of providing definitions of the numbers 0 and 1 in purely logical terms. In a well-meaning attempt to cast Frege's legacy in the most favorable light, Dummett (1981, xiii–xxv) touted his achievements in logic and its philosophical underpinnings, and downplayed his failed logicist philosophy of mathematics. Dummett (1981, xv) allowed that "Logic was, indeed, for Frege principally a tool for and a prolegomenon to the study of the philosophy of mathematics." However, if the third explanation which locates the reason for Frege's choice of the primitive quantifier symbol in his logicist account of numbers could be substantiated along the lines suggested below, that would indicate that the concept-script was not for him a mere neutral tool for studying the philosophy of mathematics but was even designed so as to serve the purposes of his logicist philosophy of arithmetic.

1 Conditionality in the Concept-Script

From a technical point of view, one notable feature of Frege's concept-script is that it has a notational device for just one binary truth-function—conditionality—and expresses the others in terms of it (with the help of the negation stroke) without introducing notational abbreviations for them. As a symbol for conditionality, Frege adopts a vertical stroke that connects two horizontal strokes; the upper and the lower horizontal stroke are respectively

followed by the consequent and the antecedent of the conditional. Thus, the conditional

$$\vdash \begin{array}{c} B \\ A \end{array}$$

corresponds in modern notation to $A \to B$. Then, using the conditional stroke, Frege (1967, sec.7) expresses conjunction ("$A \wedge B$") and disjunction ("$A \vee B$") respectively as

$$\vdash \begin{array}{c} A \\ B \end{array} \quad \text{and} \quad \vdash \begin{array}{c} A \\ B \end{array}$$

Frege (1967, sec.7) considered the idea of introducing a sign for conjunction as a primitive and defining conditionality in terms of negation and conjunction; however, he "chose the other way because [he] felt that it enables us to express inferences more simply." He says "more simply," because by taking conditionality as a basic truth-function he was able to represent any inference with more than one premise by a single rule of inference, namely modus ponens (1967, sec.6) (more on this shortly).

In "Boole's Logical Calculus and the Concept-script," Frege (1979) provides another reason for his choice of conditionality over conjunction as a primitive. He argues that since "it is a basic principle of science to reduce the number of axioms to the fewest possible," and since "[t]he more primitive signs you introduce, the more axioms you need," only the fewest possible primitive symbols should be introduced (1979, 36). For this purpose, "I must choose those with the simplest possible meanings," where a meaning is said to be simpler "the less it says" (1979, 36). Then he observes that the conditional stroke, which excludes only one possibility of assigning truth-values to the component sentences—the case of the antecedent being true and the consequent being false—says less than Boole's identity sign meaning "if and only if" and even less than Boole's multiplication sign meaning "and."

Now, as Frege (1979, 37) points out, there are four possible binary truth-functions each of which excludes only one truth-value assignment. One of them is disjunction expressed by the inclusive "or." Why choose conditionality over disjunction? Frege's (1979, 37) answer is: "because of the ease with which it can be used in inference, and because its content has a close affinity with the important relation of ground and consequent." The affinity between the content of conditionality and the "relation of ground and consequent" is evidenced by the fact that any consequence relationship between statements—such as that "B" is a consequence of "A or B" and "not A"—can be expressed

as a conditional: if *A* or *B*, then if not *A*, then *B*. This is why "an inference in accordance with any mode of inference can be reduced to [modus ponens]" (Frege 1967, sec.6). And "[s]ince it is therefore possible to manage with a single mode of inference, it is a commandment of perspicuity to do so" (Frege 1967, sec.6).

The fact that Frege chose conditionality as a primitive truth-function along with negation in the concept-script provides an explanation of why he took the universal, rather than the existential, quantifier as primitive: if the conditional sign is to be the main logical operator of a truth-functional formula, then a quantified formula with a truth-functional subformula could best be symbolized in terms of a universal quantifier. For instance, consider an I-statement of the form "Some *X* are *P*." It is standardly symbolized as "$\exists x(Xx \wedge Px)$"; but if the conjunctive subformula has to be rendered in the form of a conditional, then the whole I-statement could best be analyzed as "Not everything is such that if it is *X*, then it is not *P*," and so would be expressed in the concept-script as

(3) $\vdash \!\!\top \!\!\!\begin{array}{c} \mathfrak{a} \end{array}\!\!\!\begin{array}{l} P(\mathfrak{a}) \\ X(\mathfrak{a}) \end{array}$

Of course, it is not impossible to symbolize the I-statement in terms of an existential quantifier while keeping the conditional sign as the only binary sentential operator in its truth-functional subformula. The following will do: "$\exists x \neg (Xx \rightarrow \neg Px)$". However, (3) has an important advantage over that alternative: as is made clear by Frege's (1967, 28) diagram of "the square of the logical opposition," (3) makes explicit the contradictory relationship between the I-statement and the E-statement of the form "No *X* are *P*." The symbolization of the E-statement in the concept-script, namely

(4) $\!\!\top \!\!\!\begin{array}{c} \mathfrak{a} \end{array}\!\!\!\begin{array}{l} P(\mathfrak{a}) \\ X(\mathfrak{a}) \end{array}$

directly contradicts (3). To be sure, this contradictory relationship between the I- and the E-statement could also be made explicit using an existential quantifier by formalizing the E-statement as "$\neg \exists x \neg (Xx \rightarrow \neg Px)$." However, this formula cries out for reanalysis as "$\forall x(Xx \rightarrow \neg Px)$," that is, (4), for the sake of simplicity and naturalness.

The upshot is that if the conditional sign is employed as the only binary truth-functional operator, then the universal quantifier is better suited than

the existential quantifier to capture the logical structures of, and relationships between, quantified formulas. So Frege had a good reason to adopt the concavity as a primitive quantifier symbol in his conditionality-based concept-script.

2   Generality in the Concept-Script

Frege's 1879 monograph, *Begriffschrift*, is subtitled "*a formula language, modeled upon that of arithmetic, for pure thought.*" Arithmetic, in its narrow sense, is the theory of natural numbers, but here Frege uses the term in the sense of the theory of numbers in general. In this broad sense arithmetic includes (mathematical) analysis—or better, Analysis, with a capital "A," for distinction.[2] Analysis—the theory of functions of a real variable—involves the notions of function and variable. When Frege (1967, 6) wrote that the fact that the concept-script is modeled upon the language of arithmetic "has to do with fundamental ideas rather than with details of execution," he meant that functions and variables form the core of the design of his symbolic language of logic.

To explain in more detail, first, the concept-script replaces the traditional subject-predicate analysis of a proposition with the function-argument analysis (Frege 1967, sec.9–10). Secondly—and this is "[t]he most immediate point of contact between [his] formula language and that of arithmetic"—it adopts "the way in which letters are employed" in arithmetic (Frege 1967, 6). What Frege means by "letters" here is what mathematicians—wrongly, in Frege's (1984d, 285–88) view—refer to as variables. Arithmetic is marked partly by its use of Roman letters such as x in the formula

(5) $x^2 - 4x = x(x - 4)$.

---

2   In the titles of Frege's two books, *Foundations of Arithmetic* and *Basic Laws of Arithmetic*, "arithmetic" has this broad sense. This can be seen from Frege's remarks in *Grundlagen* §1 that "[i]n arithmetic, [...] it has been the tradition to reason less strictly than in geometry" and that "[t]he discovery of higher analysis"—namely, Leibniz's invention of the practical but less than rigorous method of infinitesimal calculus—"only served to confirm this tendency." Also, when he talks about "the great tree of the science of number as we know it, towering, spreading, and still continually growing" (1980b, sec.16), he refers to arithmetic in its broad sense, including the theory of complex numbers. *Grundgesetze* contains the beginnings of an investigation of the theory of real numbers, and there is reason to think that its planned third volume was to include a treatment of complex numbers (see Dummett 1981, 241–42).

Here x serves as a sign of generality: it indicates that the equation holds no matter what number is put for x. By incorporating in his concept-script signs of generality (as well as of functions with an arbitrary number of arguments whose value is a truth-value), Frege was able to create a symbolic language to express the full logic of quantification.

But considering that the symbolic language of arithmetic expresses generality using Roman letters alone as in (5) and does not have separate quantifier symbols, the question arises as to why Frege also introduced the concavity sign and, therewith, German letters such as $\mathfrak{a}$ in addition to Roman letters. In *Grundgesetze* he addresses the question, and says that by means of Roman letters alone it would be impossible to delimit the scope of generality for sentences such as the following (2013, sec.8):

(6) $\ \vdash_{\!\!\top}\ 2 + 3x = 5x.$

(6) admits of two different readings. First, the generality sign x can be viewed as having narrow scope with respect to the negation stroke. On this reading, (6) would express the negation of a generality, namely

(7) $\ \vdash_{\!\top}\!\!\frown^{\mathfrak{a}}\!\!\!\!\!\frown\ 2 + 3\mathfrak{a} = 5\mathfrak{a}$

which is true. Alternatively, the letter x can be viewed as having wide scope, in which case (6) expresses a false universal, namely

(8) $\ \frown^{\mathfrak{a}}\!\!\!\!\frown_{\!\top}\ 2 + 3\mathfrak{a} = 5\mathfrak{a}.$

Since it is crucial for the purposes of a logical formalism to be able to capture the difference between (7) and (8), it was necessary for Frege to introduce the concavity sign as a device for delimiting the scope of Roman letters which connote generality. Thus, although the ambiguity of (6) can be removed by "stipulating that the *scope* of a *Roman letter* is to include everything that occurs in the proposition apart from the judgment-stroke" (Frege 2013, sec.17), that is, by understanding (6) always as meaning (8), the concavity sign is still needed to express the negation of a generality such as (7).

In fact, in *Begriffschrift*, Frege (1967, sec.11) gave the same explanation of the need for the concavity sign, albeit using slightly more complicated examples. Consider the following conditional:

(9) $\begin{array}{l}\rule{1.2em}{0.5pt}\ A\\[-2pt] \rule{0.1pt}{1.4em}\!\!\frown^{\mathfrak{a}}\ X(\mathfrak{a})\end{array}$

Frege ([1967, sec.11](#)) emphasizes that (9) "does not by any means deny that the case in which $X(\Delta)$ is affirmed and A is denied does occur" for some object Δ. His point is that (9), a conditional formula, should not be confused with the following universal formula that says that such a case never occurs:

(10) $\overset{\mathfrak{a}}{\smile}\!\!\!\!\top\!\! \begin{array}{l} A \\ X(\mathfrak{a}) \end{array}$

The difference in logical content between (9) and (10) would have been lost without the concavity. So "[t]his explains why the concavity with the German letter written into it is necessary: *it delimits the scope that the generality indicated by the letter covers*" ([Frege 1967, sec.11](#)).

These considerations suggest another technical explanation of why Frege adopted the universal quantifier as a primitive. The concept-script was modeled on the symbolic language of arithmetic, and so Roman letters were used as a device to express generality. But as a result of such use of Roman letters, scope ambiguities arose, and the concavity was introduced to deal with them. Frege's adoption of the universal quantifier as a primitive was, then, a natural consequence of modeling his concept-script upon the symbolic language of arithmetic.

In order to avoid a possible misunderstanding, it should be noted that the fact that the concavity was introduced to delimit the scope of generality does not mean that it was intended to serve as a mere scope marker—a sort of punctuation sign—in such formulas as (7) and (8). That is, it would be a mistake to think that what expresses generality in (7) and (8) is the German letter $\mathfrak{a}$ in the formula "$2 + 3\mathfrak{a} = 5\mathfrak{a}$," with the concavity left to play the role of marking the scope of the letter. Frege ([1967, sec.11](#)) explains the formula "$\overset{\mathfrak{a}}{\smile}\!\!- \Phi(\mathfrak{a})$" as meaning that "whatever we may put in place of $\mathfrak{a}$, $\Phi(\mathfrak{a})$ holds," or in modern parlance, "for any value of variable $\mathfrak{a}$, Φ is true of it." This means that in the formula "$\overset{\mathfrak{a}}{\smile}\!\!- \Phi(\mathfrak{a})$," generality is expressed by the quantifier "$\overset{\mathfrak{a}}{\smile}$," not by the $\mathfrak{a}$ in "$\Phi(\mathfrak{a})$." This latter $\mathfrak{a}$ always refers to something particular—namely, a given value of the variable $\mathfrak{a}$. That is Frege's point when he writes that "the horizontal stroke to the right of the concavity is the content stroke of $\Phi(\mathfrak{a})$, and here we must imagine that something definite has been substituted for $\mathfrak{a}$" ([1967, sec.11](#)). So the concavity, with the meaning of "for any value of," is indeed a sign of generality corresponding to the modern $\forall$, and not a mere scope marker.

A related point to note is that the concavity is the only device in the concept-script to express generality. For Frege (1967, sec.11), a Roman letter is an "abbreviation" for the case where "the concavity immediately follows the judgment stroke," that is, "the content of the entire judgment constitutes the scope of the German letter." Thus, despite the fact that Roman letters precede the concavity in the order of discovery, Frege saw—rightly—the explanatory primacy of the latter over the former once he had realized that Roman letters are inadequate as a device for expressing generality due to scope ambiguities.

3   The Numbers 0 and 1

Another, different kind of explanation of Frege's adoption of the universal quantifier as a primitive could be found in the roles of universal and existential quantifiers in Frege's philosophy of arithmetic. After all, as Frege (1967, 8) acknowledged in the Preface to *Begriffsschrift*, "arithmetic was the point of departure for the train of thought that led [him] to [his] [concept-script]." Not only that; he intended "to apply it first of all to that science, attempting to provide a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems" (1967, 8). Since Frege, as a logicist, aimed to establish arithmetic as part of logic, his expressions "detailed" and "deeper" here could be understood as meaning "logical." That is, the primary applications of the concept-script were to be found in providing a logical analysis of the concepts of arithmetic and a logical foundation for its theorems. The possibility suggests itself, then, that Frege's initial attempts in that direction may have convinced him that the universal, rather than the existential, quantifier should be taken as primitive. But to support this conjecture requires evidence from Frege's early writings—early enough to have made an impact on his *Begriffsschrift* of 1879—that a logical analysis of arithmetical concepts or a logical proof of arithmetical truths compelled him to invoke the universal, rather than the existential, quantifier. Is there such evidence?

At the end of the Preface to *Begriffsschrift*, Frege (1967, 8) briefly states his future plans "to elucidate the concepts of number, magnitude, and so forth," adding that "all this will be the object of further investigations, which I shall publish immediately after this booklet." The word "immediately" here suggests that at the time of writing he was already at an advanced stage of his research about number, if not about quantity. Indeed, he reports in a letter of 1882 that "I have now nearly completed a book in which I treat the concept of number and demonstrate that the first principles of computation which up to

now have generally been regarded as unprovable axioms can be proved from definitions by means of logical laws alone" (1980a, 99). The book here referred to may well be the one that Frege (2013, IX) later said he had been forced to discard due to "internal changes within the concept-script," including changing the *Begriffsschrift* triple-bar sign ≡ for identity to the usual "equals" sign =. In *Begriffsschrift* Frege used "≡" as the identity sign (of a metalinguistic kind[3]): he presents the substitutivity principle (1967, sec.20)—that if $c \equiv d$, then if $f(c)$, then $f(d)$—as one of the two basic laws concerning the triple-bar sign along with the reflexivity principle that $c \equiv c$ (1967, sec.21). In *Grundgesetze*, Frege adopts the "equals" sign as his new identity symbol because "I have convinced myself that in arithmetic it possesses just that reference that I too want to designate" (2013, IX). That is, in *Grundgesetze*, "I use the word 'equal' with the same reference as 'coinciding with' or 'identical with'" because he has now realized that "this is also how the equality-sign is actually used in arithmetic" (2013, IX). These remarks reveal that at the time of writing *Begriffsschrift*, Frege did not think that the "equals" sign in arithmetic has the meaning of "identical with,"[4] and hence had to choose a different symbol, ≡, to denote the relation of identity. In other words, Frege, in his early period, does not seem to have regarded arithmetic as concerned with objects (as opposed to properties, relations, or functions in general), that is, those things capable of standing in the relation of identity. These considerations suggest that Frege discarded the "nearly completed" book because of his realization that numbers must be viewed as objects.

What could Frege have thought that numbers are, in his early years, if they are not objects? What could he have thought that an equality of the form "$m = n$" means if not that m is identical with n? Clues to these questions are found in *Grundlagen*. In the beginning section of Part IV, Frege (1980b, sec.55) first reminds the reader of the main lesson of Part III that "the content of a statement of number is an assertion about a concept," and then proceeds to give definitions of individual numbers which, as he puts it, "suggest themselves so spontaneously in the light of [the results of Part III]" (1980b,

---

3 Frege's (1967, sec.8) solution to the puzzle of how "$a = b$," as opposed to "$a = a$," can be informative was to take "$a \equiv b$" to talk about the names, not the objects a and b. Later he replaced it with a new solution based on the distinction between sense and meaning (1984c). For details, see Kim (2011, sec.4–5).

4 This explains why Frege (1967) uses the "equals" sign in *Begriffsschrift* only in relation to arithmetic formulas—"$(a + b)c = ac + bc$" in §1 and "$3 \times 7 = 21$" in §5—and never in non-arithmetical contexts.

sec.56). These definitions introduce the numbers 0 and 1 in the context "The number n belongs to a concept F," and so present them as properties of concepts (just as to say that wisdom belongs to Socrates is to say that wisdom is a property of Socrates). This interpretation is supported by the fact that after explaining, in §56, why those definitions must be rejected as unsatisfactory despite "suggest[ing] themselves so spontaneously,"[5] Frege (1980b, sec.57) writes that therefore "I have avoided calling a number such as 0 or 1 or 2 a *property* of a concept" (original emphasis). It is reasonable to think that this view of numbers as properties of concepts, which he presupposes in §55 as the outcome of his initial inquiry into the concept of number only to reject it in §56, was his early view of numbers (see below for more evidence); and if so, it is also reasonable to infer that in his early period he interpreted an equality of the form "$m = n$" as an equivalence of some form such as "The number m belongs to a concept $F \equiv$ the number n belongs to F," where the triple bar sign is used to indicate the "identity of content" between sentences (rather than names) as in the propositions (67) and (68) of *Begriffsschrift*.

Now, given Frege's statement in *Begriffsschrift* that he will "publish immediately after this booklet" the results of his investigation into the concept of number, it seems safe to assume that while *Begriffsschrift* was being composed, Frege may have been working on—or may even have finished (as will be evidenced below)—at least a detailed outline of the "nearly completed" book he referred to in his 1882 letter quoted above. Indeed, his remark quoted at the beginning of this section—that "arithmetic was the point of departure for the train of thought that led [him] to [his] [concept-script]"—suggests that his early attempts to give logical definitions of concepts of arithmetic and to derive some of its theorems from those definitions alone led him to devise the concept-script in the first place. It is plausible, then, that the definitions of individual numbers given in *Grundlagen* §55 were part of those early attempts of Frege to give a logicist account of arithmetic, and so predated the composition of *Begriffsschrift*.

And Frege seems to have found it necessary to invoke the universal, rather than the existential, quantifier in attempting to provide logical definitions of the numbers 0 and 1. He first observes that "[i]t is tempting to define 0" as follows (1980b, sec.55):

---

5 For an exposition and discussion of Frege's objections to the definitions in *Grundlagen* §55, see Kim (2013). For a defense and development of a theory of number based on similar definitions, see Kim (2015) and Kim (2020).

(11) The number 0 belongs to a concept F [or, more colloquially, there are 0 Fs] $=_{\text{df}}$ no object falls under the concept F [or there are no Fs].

However, he objects that (11) "seems to amount to replacing 0 by 'no,' which means the same." That is, he raises against (11) a charge of circularity that can be leveled against an attempt to define, say, "x is an ethical action" as "x is a moral action."

One might challenge this charge of circularity by maintaining that the "no" in "There are no Fs" is short for "not any," and so that the definiens of (11) should not be viewed as replacing "0" with "no" but rather as abbreviating the following:

(12) It is not the case that there exists any F [in symbols, $\neg\exists x(Fx)$].

Thus understood, (11) would seem more similar to defining "x is single" as "x is not married" than to defining "x is an ethical action" as "x is a moral action."

The problem is that an existential statement of the form "There is an F" (or, in symbols, "$\exists x(Fx)$") has the logical meaning of "There is at least one F." Frege emphasizes this fact whenever the occasion arises. In *Begriffsschrift* he observes that "If, for example, $\Lambda(x)$ means the circumstance that x is a house, then

$$\vdash\!\!\!-\!\!\stackrel{\mathfrak{a}}{\frown}\!\!\!\top \Lambda(\mathfrak{a})$$

reads 'There are houses or there is at least one house' " (1967, sec.12, n15). And a moment later he points out that the expression "some" in a statement of the form "Some M are P," "must always be understood here in such a way as to include the case 'one' as well" and that "[m]ore explicitly we would say 'some or at least one' " (1967, n16). In *Grundgesetze* Frege (2013, sec.8) is even more explicit about this, noting that the sentence

$$\vdash\!\!\!-\!\!\stackrel{\mathfrak{a}}{\frown}\!\!\!\top 2 + 3.\mathfrak{a} = 5.\mathfrak{a}$$

"says: *there is* at least one solution for the equation '$2 + 3.x = 5.x$'," and that the sentence

$$\vdash\!\!\!-\!\!\stackrel{\mathfrak{a}}{\frown}\!\!\!\top \mathfrak{a}^2 = 1$$

has the meaning of "*there is* at least one square root of 1." In §13, he notes that "the plural ['some'] is not to be understood as requiring that there must be more than one" but as meaning "there is at least one."[6] Thus, given this

---

6 For similar remarks, see also Frege (1984a, 152–53; 1979, 14, 21, 61; and 1980a, 101–2).

fact that an existential statement has the meaning of "there is at least one ...," taking the existential quantifier as primitive and defining the number 0 as in (13)

(13) The number 0 belongs to a concept $F =_{df}$ it is not the case that there is at least one F

would have exposed Frege to the charge of defining 0 in terms of the number word "one" and so of smuggling in an arithmetical concept while attempting to give logical definitions of arithmetical concepts.

It is for that reason that Frege (1980b, sec.55) proposes instead that "[t]he following formulation is therefore preferable: the number 0 belongs to a concept, if the proposition that a does not fall under that concept is true universally, whatever a may be." The proposal is, in effect, to define the number 0 in terms of the universal quantifier as follows:

(14) The number 0 belongs to a concept $F =_{df}$ all things are non-Fs [in symbols, $\forall x\neg(Fx)$].

And it is also for that same reason that Frege (1980b, sec.55) suggests the following, rather awkward definition of the number 1:

(15) The number 1 belongs to a concept $F =_{df}$ not all things are non-Fs and if any things are Fs, then they are the same [in symbols: $\neg\forall x\neg(Fx) \land \forall x\forall y((Fx \land Fy) \rightarrow x = y)$].

This definition could have been made simpler by replacing "not all things are non-Fs [$\neg\forall x\neg(Fx)$]" by "there is an F [$\exists x(Fx)$]." However, that option was not open to Frege, for it meant, from his point of view, that the number 1 was defined in terms of the word "one," which means the same.

The realization that Frege was compelled to define the number 0 in terms of the universal quantifier as in (14) enables an understanding of his otherwise rather puzzling thesis about existence advanced in §53 of *Grundlagen*, namely that

> Affirmation of existence is in fact nothing but denial of the number nought.

This might be called the *Existence-Zero thesis*, or EZ for short. EZ would seem puzzling considering how Frege (1980b, sec.74) ultimately defined the number 0:

(16) 0 =$_{df}$ the number of objects that are not self-identical.

If EZ were based on this definition of the number 0, then what it says could be formulated thus:

(17) There exists an F[7] ↔ the number of Fs ≠ the number of objects that are not self-identical.

But (17) does not say the same as EZ. To see this, note that for Frege (1980b, sec.73), the right-hand side of (17) says that the concept F is not equinumerous to the concept *non-self-identical object*, where two concepts G and H are said to be equinumerous just in case there is a one-one correlation between the Gs and the Hs. So what (17) says is in fact the following:

(18) There exists an F ↔ ¬(there is a one-one correlation between the Fs and the non-self-identical objects).

This biconditional does hold: if there exists no F, then trivially there will be a one-one correlation between the Fs and the non-self-identical objects, and *vice versa*. However, the right-hand side of (18) contains the expression "there is a one-one correlation" which is of the form "there exists an F," that is, of the same form as the left-hand side. Thus, (18) cannot be viewed as offering an explanation of what existence is, whereas that is what EZ is supposed to do: it is supposed to explain the notion of existence in terms of the number 0.

The expression "nothing but" used in the above statement of EZ indicates that for Frege, the relationship between affirmation of existence and denial of the number 0 holds by definition, that is, that EZ is true by virtue of the meaning of "exists." That would make sense if, at the time of writing *Grundlagen* §53, Frege thought that the number 0 could be defined as in (14). For, then, the following two biconditionals would hold:

(19) $\exists x(Fx) \leftrightarrow \neg\forall x\neg(Fx) \leftrightarrow \neg$ (the number 0 belongs to F).

The first biconditional holds because, as noted above, a statement of the form "There is at least one F" or "$\exists x(Fx)$" is expressed in Frege's concept-script as "$\vdash_{\top}\!\mathfrak{a}_{\top} F(\mathfrak{a})$," or in modern notation, "$\neg\forall x\neg(Fx)$"; and the second biconditional is a corollary of (14). Thus, (19) is a simple consequence of two definitions, and to that extent, could be regarded as a definitional truth itself. Hence, affirmation of existence—"$\exists x(Fx)$"—is nothing but denial of the number 0—"$\neg$ (the number 0 belongs to F)."

Incidentally, the fact that EZ makes better sense when the number 0 is understood in the sense of (14) suggests that *Grundlagen* §53, where the thesis is advanced, reflects his early view of numbers as properties of concepts rather than his mature view of numbers as objects. This is further supported by his remarks in §53 that "existence is analogous to number" and that "existence is a property of concepts." So when Frege wrote at the beginning of §56 that the definitions in §55 "suggest themselves so spontaneously in the light of our previous results, that we shall have to go into the reasons why they cannot be reckoned satisfactory," he was renouncing his own early view of numbers as properties of concepts.

One might object that Frege's fundamental insight that a statement of number contains an assertion about a concept, which was first put forward in §46 of Part III and then reiterated at the beginning of §55 as the main lesson of Part III, continued to be upheld even in *Grundgesetze* where Frege (2013, IX) calls it "[t]he basis for my results," and that this suggests that there is no discontinuity between Frege's view of number in Part III of *Grundlagen* and his later view. But that is no objection, for that insight itself is compatible with both the early view of numbers as properties of concepts and the later view of numbers as objects. In fact, the very reason that the insight is compatible with the latter is that Frege's number-objects, as extensions of concepts, are proxies for properties of concepts.

One might also object that since in *Grundlagen* §38, Frege draws the distinction between proper names and concept words, and classifies the word "one" as a proper name, and since in §51, he declares that "The business of a general concept word"—a word "used with the indefinite article or in the plural without any article"—"is precisely to signify a concept," he must have already believed in Part III of *Grundlagen* that number words such as "one" refer to objects. But this objection assumes, wrongly, that in the earlier parts of *Grundlagen* Frege already upheld his (1984b) later dichotomy between expressions referring to objects, namely proper names, and those referring to concepts, namely predicates. Frege indeed says in §51 of Part III that "when

conjoined with the definite article or a demonstrative pronoun" "[a general concept word] can be counted as the proper name of a[n object]."[8] However, in this context, "general concept word" means an expression for a first-level concept such as "satellite of the Earth." As is clear from his ensuing remark that "It is to concepts of just this kind (for example, satellite of the Earth) that the number 1 belongs," the word "number," when combined with the definite article, is meant to refer not to an object[9] but to a property that belongs to first-level concepts. In other words, since numbers are second-level properties, the word "number," when conjoined with "the," refers to a second-level property, and so does not behave like a general concept word which refers to an object when preceded by "the." Also, recall in this connection the fact that when Frege (1980b, sec.55) gives definitions of individual numbers conceived as properties of concepts, he does so in the context "The number *n* belongs to a concept *F*," apparently thinking that expressions of the form "the number *n*" refer to properties of concepts. So Frege's (1980b, sec.57) realization that "In the proposition 'the number 0 belongs to the concept *F*,' 0 is only an element in the predicate"—namely the second-level predicate "the number 0 belongs to"—and hence cannot denote a second-level property in its own right represents a profound break from his earlier view of number words as referring to second-level properties (despite being proper names).

In light of the above considerations it seems reasonable to hypothesize that the 1884 *Grundlagen* was not conceived and written in its entirety in response to Carl Stumpf's suggestion, in a letter dated September 9, 1882, of "explain[ing] your line of thought first in ordinary language" (Frege 1980a, 172). It is more likely that Frege set out to rewrite in ordinary language the symbolic parts of his "nearly completed" "book in which I treat the concept of number." And, while doing so, he may have come up with the objections raised in *Grundlagen* §56 to his early view of numbers as properties of concepts, and been led to the conclusion that numbers must be objects instead. The first three parts of *Grundlagen* could be the parts of the discarded book that were salvaged.

---

8 In the original, the word "thing [*Ding*]" is used, because the comment was made in response to Schröder's claim that abstraction "has the effect of turning what was the name of the thing into a concept applicable to more than one thing" (Frege 1980b, sec.50).

9 Frege (1980b, sec.45) describes the word "one" as "the proper name of an object of mathematical study," but the word "object" here does not necessarily mean what it means when he (1980b, sec.57) concludes that numbers are objects (as opposed to properties or relations).

The conjecture that the first three parts of *Grundlagen* contain Frege's early reflections on number has direct textual support in the "Notes for Ludwig Darmstaedter":

> I started out from mathematics. The most pressing need, it seemed to me, was to provide this science with a better foundation. I soon realized that number is not a heap, a series of things, nor a property of a heap either, but that in stating a number which we have arrived at as the result of counting we are making a statement about a concept. [...] The logical imperfections of language stood in the way of such investigations. I tried to overcome these obstacles with my concept-script. In this way I was led from mathematics to logic. (1979, 253)

The third sentence in this quote reads like a quick summary of the first three parts of *Grundlagen*. Thus, if the narrative is to be believed, Frege had obtained all the results of those parts of *Grundlagen*, including his fundamental insight about the content of a statement of number, before he even conceived the idea of a concept-script. The concept-script was later invented as a means to overcome the obstacles he encountered while carrying out the further investigations, using ordinary language, into analysis of arithmetical concepts and proof of arithmetical truths. So Frege's claim in the 1882 letter that "I have now nearly completed a book" on number could be understood as saying that those further investigations that caused him difficulties due to the "logical imperfections of language" have been nearly completed with the help of the newly invented concept-script. The nontechnical parts of the book—Parts I–III of *Grundlagen*—had been completed before its invention.

To return to the main issue of this section, Frege's goal of providing analysis of arithmetical concepts in purely logical terms meant that he could not adopt the existential quantifier as a primitive. Since existential statements—including those of the form "Some *M* are *P*"—have the meaning of "there is at least one ...," Frege needed to paraphrase them so as to avoid making reference to the numerical notion of one. This he (1980b, sec.55) achieved by defining the number 0 in terms of a universal negative ("$\forall\neg$"), which allowed him to paraphrase an existential statement in purely logical terms as a negative universal negative ("$\neg\forall\neg$"), that is, as a "denial of the number nought" (1980b, sec.53). Thus, the fact that for Frege, affirmation of existence is nothing but denial of the number 0 is explained by, and hence adds support

to, the conjecture that he was forced to adopt the universal quantifier as a primitive by his felt need to avoid using an existential quantifier in his definitions of the numbers 0 and 1. Of course, in the end—in *Grundlagen* §56—he abandoned the definitions given in §55, including (14) and (15), and opted to define explicitly each individual number as the number of *F*s for some suitable concept *F* as illustrated in (16). However, the point remains that the definitions of *Grundlagen* §55 along with the thesis EZ of §53 are likely to have been part of his early reflections on number and so to have formed "the train of thought that led [him] to [his] [concept-script]" (Frege 1967, 8), including the decision to adopt the universal quantifier as a primitive.

4 Conclusion

The preceding sections have provided three possible explanations—two technical and one philosophical—of Frege's adoption of the universal quantifier as a primitive in his concept-script. This concluding section briefly discusses their relative merits.

As noted at the beginning of this paper, Frege nowhere says anything about why he took the universal, rather than the existential, quantifier as primitive. To that extent one could not reach a definite conclusion as to which of the three possible reasons, if any, was the real reason for Frege's adoption of the universal quantifier as a primitive. Perhaps it is more likely than not that to varying degrees all three of them contributed to and helped cement his decision.

That said, the question could be raised as to which of the three explanations provides the strongest justification for taking the universal quantifier as primitive. And from this point of view, the most satisfactory explanation seems to be the third one. Given the interdefinability of the universal and existential quantifiers, the first two explanations alone do not seem sufficient to make unavoidable the use of the universal quantifier as a primitive. Admittedly, it would have been unnatural and inefficient to use the existential quantifier as a primitive considering that the concept-script has conditionality as the sole binary truth-function; still, it was not an impossibility.

By contrast, the philosophical explanation shows that Frege had no alternative but to adopt the universal quantifier as a primitive. For, given his recurring theme that the existential quantifier involves the notion of "at least one," using it as a primitive would have conflicted with his goal of analyzing

arithmetical concepts, especially the concepts of 0 and 1, in purely logical terms.

Relatedly, this explanation has an additional, decisive advantage: it renders understandable Frege's otherwise puzzling silence on the interdefinability of the universal and existential quantifiers. As noted in section 1, he addresses in detail the interdefinability of conditionality and conjunction and explains why he chose the former as a primitive (1967, sec.7). Thus, as Macbeth (2005, 4) rightly points out, "Had he thought that there were two logically admissible quantifiers usable for the expression of generality, [...] he would have said so." But he did not say so, and this fact indicates that he did not think that the universal and existential quantifiers are equally admissible. And one can understand why given the third explanation for Frege's adoption of the universal quantifier as a primitive. Taking the existential quantifier as an equally admissible primitive would have amounted to allowing into logic what is apparently an arithmetical notion—the notion of one—which is unacceptable from his logicist viewpoint.*

Joongol Kim
Sogang University
joongolk@sogang.ac.kr

References

Cook, Roy T. 2013. "How to Read *Grundgesetze*." In *Basic Laws of Arithmetic*, A-1-A-42. Oxford: Oxford University Press. Translated and edited by Philip A. Ebert and Marcus Rossberg, with Crispin Wright.

Dummett, Michael A. E. 1981. *Frege: Philosophy of Language*. 2nd ed. Cambridge, Massachusetts: Harvard University Press.

Frege, Gottlob. 1884. *Die Grundlagen der Arithmetik: eine logisch-mathematische Untersuchung über den Begriff der Zahl*. Breslau: Wilhelm Koebner.

———. 1891. *Function und Begriff. Vortrag, gehalten in der Sitzung vom 9. Januar 1891 der Jenaischen Gesellschaft für Medizin und Naturwissenschaft*. Jena: Hermann Pohle.

———. 1892a. "Über Begriff und Gegenstand." *Vierteljahrsschrift für wissenschaftliche Philosophie* 16: 192–205.

———. 1892b. "Über Sinn und Bedeutung." *Zeitschrift für Philosophie und philosophische Kritik NF* 100: 25–50.

———. 1893. *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*. Vol. I. Jena: Hermann Pohle.

———. 1904. "Was ist eine Funktion?" In *Festschrift Ludwig Boltzmann gewidmet zum sechzigsten Geburtstage, 20. Februar 1904*, edited by Stefan Meyer, 656–66. Leipzig: Johann Ambrosius Barth.

———. 1967. "*Begriffsschrift*, a Formula Language, Modeled on That of Arithmetic, for Pure Thought." In *From Frege to Gödel: A Source Book in Mathematical Logic 1879-1931*, edited by Jan van Heijenoort, 1–82. Cambridge, Massachusetts: Harvard University Press. Translated by Stephan Bauer-Mengelberg.

———. 1969. *Nachgelassene Schriften*. Hamburg: Felix Meiner Verlag. Edited by Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach.

———. 1976. *Wissenschaftlicher Briefwechsel*. Hamburg: Felix Meiner Verlag.

———. 1979. *Posthumous Writings*. Oxford: Basil Blackwell Publishers. Edited by Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach; translation of Frege (1969) by Peter Long and Roger White.

———. 1980a. *Philosophical and Mathematical Correspondence*. Oxford: Basil Blackwell Publishers. Edited by Gottfried Gabriel, Hans Hermes, Friedrich Kambartel, Christian Thiel and Albert Veraart; translation of (Frege 1976) by Hans Kaal and abridged by Brian McGuinness.

———. 1980b. *The Foundations of Arithmetic*. 2nd ed. Evanston, Illinois: Northwestern University Press. Translation of Frege (1884) by J.L. Austin.

———. 1984a. "Function and Concept." In *Collected Papers on Mathematics, Logic, and Philosophy*, 137–56. Oxford: Basil Blackwell Publishers. Translation of Frege (1891) by Peter Geach.

———. 1984b. "On Concept and Object." In *Collected Papers on Mathematics, Logic, and Philosophy*, 182–94. Oxford: Basil Blackwell Publishers. Translation of Frege (1892a) by Peter Geach.

———. 1984c. "On Sense and Meaning." In *Collected Papers on Mathematics, Logic, and Philosophy*, 157–77. Oxford: Basil Blackwell Publishers. Translation of Frege (1892b) by Max Black.

———. 1984d. "What Is a Function?" In *Collected Papers on Mathematics, Logic, and Philosophy*, 285–92. Oxford: Basil Blackwell Publishers. Translation of Frege (1904) by Peter Geach.

———. 2013. *Basic Laws of Arithmetic*. Oxford: Oxford University Press. Translation and edition of Frege (1893) by Philip A. Ebert and Marcus Rossberg, with Crispin Wright.

KIM, Joongol. 2011. "Frege's Context Principle: An Interpretation." *Pacific Philosophical Quarterly* 92 (2): 193–213. doi:10.1111/j.1468-0114.2011.01391.x.

———. 2013. "What Are Numbers?" *Synthese* 190 (6): 1099–1112. doi:10.1007/s11229-011-9883-y.

———. 2015. "A Logical Foundation of Arithmetic." *Studia Logica (Poznan Panstwowe Wydawnictowo Naukowe)* 103 (1): 113–44. doi:10.1007/s11225-014-9551-6.

———. 2020. "The Adverbial Theory of Numbers: Some Clarifications." *Synthese* 197 (9): 3981–4000. doi:10.1007/s11229-018-01911-1.

Kneale, William C., and Martha Kneale. 1962. *The Development of Logic.* Oxford: Oxford University Press.

Macbeth, Danielle. 2005. *Frege's Logic.* Cambridge, Massachusetts: Harvard University Press.