David Armstrong on the Metaphysics of Mathematics

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This paper has two components. The first, longer component (sec. 1-6) is a critical exposition of Armstrong's views about the metaphysics of mathematics, as they are presented in *Truth and Truthmakers* and *Sketch for a Systematic Metaphysics*. In particular, I discuss Armstrong's views about the nature of the cardinal numbers, and his account of how modal truths are made true. In the second component of the paper (sec. 7), which is shorter and more tentative, I sketch an alternative account of the metaphysics of mathematics. I suggest we insist that mathematical truths have physical truthmakers, without insisting that mathematical objects themselves are part of the physical world.

A prime number p is a "Sophie Germain prime" if $(2p + 1)$ is also prime. It is conjectured that there exist infinitely many Sophie Germain primes. I don't know whether this conjecture is true, but what I do know is that there exist *some* Sophie Germain primes: 2 is an example; 3 is another; $(2,618,163,402,417 \times 2^{1290000} - 1)$ is a third, or so I am told. Now it is obvious that every Sophie Germain prime is a number; and it follows that there exist some numbers—or so it seems.

But what kind of a thing is a number?

This is a difficult question, but one point at least seems clear: numbers and other mathematical entities are "abstract," in the sense that they have no causal powers and no location in spacetime. We are told that the number zero was discovered in India, but it would be a mistake to go to India *now* to look for it—and not because it has subsequently been moved. You can't trip over the number three. The polynomial $(x^2 - 3x + 2)$ can be split into two factors, $(x - 2)$ and $(x - 1)$, but not by firing integers at it in a particle accelerator. The empty set has no gravitational field. And so on.

And so, to labour the point, it seems that some abstract entities exist.

And yet David Armstrong began his last book by endorsing what he called "naturalism":

> I begin with the assumption that all that exists is the space-time world, the physical world as we say. [...] [This] means the rejection of what many contemporary philosophers call "abstract objects," meaning such things as numbers or Platonic Forms or classes, where these are supposed to exist "outside of" "or"extra to" space-time. (2010, 1)

Despite this naturalism, Armstrong did not reject any part of mainstream mathematics. Indeed, he insisted that the truths of orthodox pure mathematics are necessary and *a priori* (2010, Ch. 12).

In this paper, I explore Armstrong's attempt to reconcile his denial of abstract entities with his commitment to orthodox mathematics. To be more specific, the paper has three goals.

1. Armstrong wrote a vast amount on mathematics, and this writing is spread among many papers and books. Some of this work is complex, and Armstrong changed his mind on certain important questions. My first goal in this paper is to describe—clearly, briefly, and in one place — Armstrong's mature views on the metaphysics of mathematics, including relevant aspects of his work on the metaphysics of modality. To prevent the discussion from sprawling, I focus particularly on Armstrong's account of cardinal number, as it is presented in his last two books: *Truth and Truthmakers* (2004, "T&T") and *Sketch for a Systematic Metaphysics* (2010, "SSM"). Sec. 1 and section 2 describe Armstrong's views about the cardinal numbers; section 3 – section 6 focus on modality.
2. My second goal is to present some novel—and, I believe, definitive—objections to Armstrong's views on the metaphysics of mathematics. These objections are presented in section 5 and section 6.
3. My third goal is to recommend a different way of thinking about the metaphysics of mathematics—an approach which will, I hope, appeal to people who admire Armstrong's work. Briefly, I will suggest that we insist that every mathematical truth has a truthmaker in the physical world, without also insisting that mathematical objects themselves are physical things. The proposal is presented in more detail in section 7.

1 Cardinal Numbers as Concrete Entities

The claim that numbers have no spatial location is familiar to metaphysicians, but it sometimes comes as a surprise to students. "But there are three pens on my desk *right now!*" they exclaim, implying that the number three itself is within arm's reach. According to Armstrong, the surprised students are on to something.

While he rejected "Platonic forms" which exist outside spacetime, Armstrong did believe that there are properties which exist *within* the particulars that instantiate them (SSM, ch. 2). For Armstrong, there exists a property *is red* which exists within London buses, ripe tomatoes, and male cardinals. As he sometimes put it, properties are "immanent" rather than "transcendent". He inferred that all properties are instantiated. Uninstantiated properties have no place in spacetime, and so no place in Armstrong's philosophical system (SSM, 15–16). He made the same claim about relations (SSM, 23).

For Armstrong, a cardinal number is a relation between a particular and a property. Specifically, the cardinal number κ is a relation that a particular *x* bears to a property P just in case *x* has, as mereological parts, exactly κ particulars which instantiate P. A normal octopus bears the *one* relation to the property *is an octopus,* and the *eight* relation to the property *is a limb.* The mereological sum of two normal octopuses bears the *two* relation to the property *is an octopus* and the *sixteen* relation to the property *is a limb.* And so on.[1, 2]

On Armstrong's view, then, the surprised student is correct to think that cardinal numbers exist within the "spacetime world". It turns out, then, that one can coherently maintain that numbers exist and that there are no abstract entities.

This is a striking result—and yet, a problem looms. As I said, Armstrong insisted that properties and relations exist only when they are instantiated. On this view, the number $10^{10^{10}}$ exists only if the spacetime world happens to

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[1] Armstrong's theory of cardinal number is closely related to that presented in Kessler (1980). Simons (1982) raises some important objections to Kessler's account. I believe that Armstrong's theory is not vulnerable to Kessler's objections, though I will not pursue the issue here.

[2] Note that the property P will in many cases be a "second-rate property" rather than a genuine universal (SSM, 19). For example, there are seven medium-sized red spoons in my apartment. As Armstrong would put it, the fusion of the contents of my apartment bears the *seven* relation to the property *medium-sized red spoon.* But Armstrong would surely deny that *medium-sized red spoon* is a universal. Armstrong understood that we need "second-rate" properties in cases like this (T&T, 72).

contain $10^{10^{10}}$ particulars. And it is far from clear that the spacetime world is
that large. I call this "the problem of size".

It is tempting to reply to this objection by insisting that space is infinitely
divisible. Discussing the problem of size as it arises for Aristotle, Jonathan
Barnes writes:

> Physical objects are, in Aristotle's view, infinitely divisible. That
> fact ensures that, even within the actual finite universe, we shall
> always be able to find a group of k objects, for any k [...] If the
> universe consisted simply of a single sphere, it would also contain
> two objects (two hemispheres), three objects (three third-spheres)
> and so on. We shall never run short of numbers of things [...].
> (1985, 122)

This cannot be considered a satisfactory solution to *Armstrong's* problem,
however. For one thing, it is far from clear that Aristotle was correct in thinking
that space is infinitely divisible: those who study quantum gravity have been
known to speculate that space is in fact discrete. And so the proposed solution
is somewhat "hostage to fortune." More importantly, even if the proposed
account does secure the existence of large finite numbers such as $10^{10^{10}}$,
it still leaves the existence of transfinite cardinals open to doubt. Standard
mathematical descriptions of spacetime (which do imply infinite divisibility)
entail that the set of spacetime points has cardinality $\beth_1$. Such accounts
leave it unclear whether larger cardinal numbers (e.g. $\beth_2$, $\beth_3$, or even $\beth_\omega$) are
instantiated in the physical world—and yet these larger cardinals are very
much a part of orthodox mathematics.[3]

In passing, I note that Armstrong faced a problem of size in his account of
set theory too. While Armstrong identified cardinal numbers with relations, he
identified sets with individuals. The central claim in Armstrong's account
was David Lewis's "brilliant insight" (T&T, 120): the mereological parts of a
set are precisely its non-empty subsets. For example, Lewis's claim implies
that the mereological proper parts of {a, b, c} are {a}, {b}, {c}, {a, b}, {b, c}, and {c,
a}. This implies that every singleton set is mereologically simple. Now the set
theorists tell us that for each cardinal number κ, there are at least κ singletons.
Thus, Armstrong is stuck with the claim that, within the spacetime world,
there are least κ mereologically simple individuals, for each κ. And this seems
highly doubtful. Perhaps one could plausibly argue that there $\beth_1$ mereological

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3 For an introduction to the mathematics of these enormous numbers, see Yarnelle (1964).

simples by saying that each spacetime point is mereologically simple and there are $\beth_1$ of those. But it is hard to see any justification for the claim that there are $\beth_2$, or $\beth_3$, or even $\beth_\omega$ mereological simples in the spacetime world. Once again, the physical universe seems to be too small to accommodate the ontology of mathematics.[4]

2 Armstrong's Possibilism

Armstrong was aware of the problem of size. He responded by claiming that larger cardinal numbers exist *in posse* though not *in esse:*

> A Platonist will solve the problem [of size] by postulating unin-stantiated numbers. [...] [However,] my own hard choice [...] is to accept a deflationary doctrine of what it is for a mathemati-cal entity to exist. Plenty of mathematical structures exist in a straightforward sense, because they are instantiated. We can call them *empirical* mathematical structures. [...] But mathematical existence itself, I suggest, should be reckoned as something less. A mathematical entity exists if and only if it is *possible* that there should be instantiations of that structure. (T&T, 117)

Armstrong called this *possibilism*. The doctrine is not easy to interpret. In the above passage, Armstrong seems to suggest that a mathematical entity exists provided that it *could* be instantiated — even if it is in fact not instantiated. However, in this same passage, Armstrong *contrasts* his possibilism with the "Platonist" claim that there are uninstantiated mathematical entities.

The following quotation gives us a clue about what Armstrong meant:

> We say '7 + 5 = 12', but this can rendered more transparently, though more boringly, as ⟨Necessarily, if there are seven things, and five further things, then the sum of these things are twelve things⟩. (T&T, 101)

What this passage suggests is that, for Armstrong, while the sentence "7 + 5 = 12" appears to describe a relation among three mathematical entities (viz. the numbers seven, five and twelve) it is in fact a generalization about pluralities of marbles, pebbles, sticks, or whatever. More generally, the possibilist main-tains that while pure mathematics appears to describe a domain of special

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4 For a more thorough discussion of this point, see Rosen (1995).

mathematical entities, it in fact consists of *modal* statements—statements about what is necessary or possible.

Now Armstrong did not develop this proposal systematically; instead, he endorsed Geoffrey Hellman's modal structuralism:

> I recognize, of course, that asserting here this […] doctrine of mathematical existence is to a degree a matter of hand-waving. I have not the logico-mathematical grasp to defend it in any depth. That has been done, in particular by Geoffrey Hellman. (T&T, 117)

The reader who wants a thorough discussion of modal structuralism should consult Hellman (1989). For now, a back-of-the-envelope summary will be sufficient. The modal structuralist claims that the theory of the natural numbers is not a description of some particular sequence of entities; rather, the theory concerns *all possible* models of the Peano axioms.[5]

For example, when a mathematician asserts that there are infinitely many prime numbers, what is really meant is something like this:[6]

> It is necessary that, in any model of the Peano axioms, the domain contains infinitely many prime elements.

Notice that this statement does not imply that there is a model of the Peano axioms somewhere in spacetime.

When a modal structuralist mathematician asserts that, necessarily, every model of the Peano axioms contains infinitely many prime elements, she will of course wish to rule out the suggestion that this is true "vacuously"—that is, simply because models of the Peano axioms are *impossible.* Thus, a modal structuralist mathematician will claim that models of the Peano axioms are possible.

Hellman discusses in some detail how to extend this approach beyond number theory, and into applied mathematics. We need not look into these details. For our purposes, the key point is that the appeal to modal structuralism allows Armstrong to say that "$10^{10^{10}}$ is even" is true (when properly interpreted)

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5 The Peano axioms are the standard axioms in the theory of the natural numbers. Among them are such claims as "Zero is a number", and "If x is any natural number, $x + 0 = 0$".

6 The version of modal structuralism that I so quickly sketch here is hermeneutic rather than revolutionary (for this distinction, see J. P. Burgess and Rosen 1997, 6–7). This is, I think, the correct interpretation of Armstrong's position. For Hellman's position, see (1998).

without committing himself to the questionable thesis that $10^{10^{10}}$ physical objects exist.[7]

The attractions of the approach are obvious, but Armstrong's possibilism brings with it a new problem. Armstrong was a truthmaker maximalist — he believed that every true proposition has a "truthmaker," that is, an entity in the spacetime world which is sufficient (and perhaps more than sufficient) to explain the truth of the proposition.[8] Armstrong was thus stuck with the formidable task of identifying truthmakers for complex modal truths like those described above. It is my contention that Armstrong did not succeed at this task, as I shall explain in the next three sections.[9]

3 Armstrong's Entailment Principle

Before we look at Armstrong's discussion of truthmaking and modality, we must consider his Entailment Principle, which is crucial to his account. Some notation will be helpful: I will put a sentence between angled brackets to represent the corresponding proposition. For example, ⟨Sam is dancing⟩ is the proposition that Sam is dancing. Here is the Entailment Principle, as it is formulated in *Sketch for a Systematic Metaphysics*:

> *The Entailment Principle (SSM Version).* If α entails β, then any truthmaker for ⟨α⟩ must be a truthmaker for ⟨β⟩ too. (SSM, 65–66)

For example, since φ entails $\neg\neg\varphi$, it is a consequence of Armstrong's entailment principle that any truthmaker for ⟨φ⟩ must also be a truthmaker

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7 Hellman's modal structuralism involves second-order quantification, and it is worth thinking about how such quantifiers should be interpreted within Armstrong's metaphysical system. One approach is to say that the second-order variables range over properties (including "second-rate" properties—see footnote 2). Some restriction of the usual comprehension axiom will be needed to accommodate Armstrong's contention that there are no uninstantiated properties. For a version of modal structuralism that does not require second order quantification, see Berry (2018).

8 The parenthetical "and perhaps more than sufficient" is there to indicate that Armstrong's was an *inexact* conception of truthmaking, to use Kit Fine's terminology—see (2017).

9 Fox (1987) endorses a *purely modal* conception of truthmaking. According to Fox, T is a truthmaker for p just in case it is necessary that if T exists then p is true. On this approach, it is *easy* to identify truthmakers for purely mathematical truths. Since the truths of pure mathematics are necessary, given Fox's purely modal conception of truthmaking, *anything whatever* is a truthmaker for any purely mathematical truth.

Armstrong himself vigorously rejected this approach, insisting that truthmakers must be *relevant* to the propositions they make true (T&T, 11). For more on this theme, see Cameron (2018).

for $\langle\neg\neg\varphi\rangle$. In this example, the Entailment Principle is plausible. However, there is an important objection to this formulation of the principle. As Restall (1996) has pointed out, this simple version of the Entailment Principle conflicts with a popular and appealingly simple (though not undisputed) account of truthmaking and disjunction:

> *The Disjunction Principle. T* makes true the proposition $\langle\varphi \vee \psi\rangle$ if and only if *T* makes true $\langle\varphi\rangle$, or *T* makes true $\langle\psi\rangle$, or both.

To see the conflict, consider the following argument:

> Let φ and ψ be any two true sentences, and suppose that *T* is a truthmaker for $\langle\varphi\rangle$. Then since φ entails $(\psi \vee \neg\psi)$, *T* must also be a truthmaker for $\langle\psi \vee \neg\psi\rangle$. By the Disjunction Principle, *T* must be a truthmaker either for $\langle\psi\rangle$ or for $\langle\neg\psi\rangle$. But by hypothesis, $\langle\psi\rangle$ is true so $\langle\neg\psi\rangle$ is false and so $\langle\neg\psi\rangle$ has no truthmakers. So *T* must be a truthmaker for $\langle\psi\rangle$.

This little argument appears to show for any two true sentences φ and ψ, any truthmaker for $\langle\varphi\rangle$ is also a truthmaker for $\langle\psi\rangle$—a result which completely trivializes truthmaker theory.

In *Truth and Truthmakers,* Armstrong gives a more sophisticated version of the Entailment Principle which is not subject to the same objection:

> *The Entailment Principle, (T&T Version).* If α entails* β, then any truthmaker for $\langle\alpha\rangle$ must be a truthmaker for $\langle\beta\rangle$ too. (T&T, 10)

Here, entailment* is some non-classical entailment relation, to be specified. By insisting that φ need not entail* $(\psi \vee \neg\psi)$, we can maintain a version of the Entailment Principle without having to conclude, absurdly, that all propositions expressed by true sentences have the same truthmakers.[10]

We have seen that the "SSM version" of the Entailment Principle conflicts with the Disjunction Principle, and that one can maintain the Disjunction Principle by endorsing the "T&T version" instead. If a truthmaker theorist wishes instead to maintain the simpler, "SSM version" of the Entailment

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10 It is not easy to say exactly what entailment* is. Restall (1996) has proposed that entailment* "is nearly, but not quite, the first degree entailment of relevant logic". Linnebo (2017) has suggested that entailment* includes first-order intuitionistic entailment without identity. Thankfully, we need not settle this question here.

Principle, she may choose to reject the Disjunction Principle—and indeed philosophers have presented *independent* reasons for rejecting this principle.[11] We need not settle this dispute here. Suffice it to say that appeals to the "SSM version" of the Entailment Principle are subject to dispute.[12]

4 Armstrong on Truthmaking and Possibility

Having briefly looked at the Entailment Principle we are ready to consider Armstrong's account of truthmaking and modality. Let's start with possibility. Suppose that the sentence φ expresses a contingently true proposition; what then are the truthmakers for $\langle\Diamond\varphi\rangle$ and $\langle\Diamond\neg\varphi\rangle$?

$\langle\Diamond\varphi\rangle$ is comparatively straightforward. Since $\langle\varphi\rangle$ is true, Armstrong argued, it must have a truthmaker, T. Since φ entails $\Diamond\varphi$, T will be a truthmaker for $\langle\Diamond\varphi\rangle$ as well, by the Entailment Principle.[13]

$\langle\Diamond\neg\varphi\rangle$ is rather more difficult. Armstrong introduced his "possibility principle" (T&T, 84) to deal with the problem:

> *Possibility Principle.* If $\langle\varphi\rangle$ is a contingent truth and T is a truthmaker for $\langle\varphi\rangle$, then T is a truthmaker for $\langle\Diamond\neg\varphi\rangle$.

The principle is not attractive on its face. As Pawl (2010) has pointed out, Armstrong's being legged is a truthmaker for ⟨Someone has legs⟩, but it is hardly plausible that Armstrong's being legged is a truthmaker for ⟨Possibly, nobody has legs⟩. But Armstrong claimed that the Possibility Principle is a consequence of the Entailment Principle. He presented the following argument (T&T, 84, notation slightly modified):

1. T is a truthmaker for $\langle\varphi\rangle$. (Assumption)
2. $\langle\varphi\rangle$ is contingent. (Assumption)
3. $\langle\varphi\rangle$ entails $\langle\Diamond\neg\varphi\rangle$. (From 2, and the nature of the contingency of propositions)

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11 For discussion of the Disjunction Principle, see Rodriguez-Pereyra (2006) and López de Sa (2009).

12 Why did Armstrong give these two different versions of the Entailment Principle, in books written in the same decade? My hypothesis is that T&T contains Armstrong's preferred formulation of the Entailment Principle, and that the version in SSM (a much shorter, easier work) is a simplification.

13 It is not entirely straightforward that this application of the Entailment Principle is correct. While it is clear that φ entails $\Diamond\varphi$, it is perhaps not so clear that φ entails* $\Diamond\varphi$. We can let this point slide, however, because there are much more serious objections to Armstrong's position, as we shall see.

4. *T* is a truthmaker for $\langle\Diamond\neg\varphi\rangle$. (From 1, 3 and the Entailment Principle)

As Armstrong later recognized, this argument is fallacious. The error is in step (3): in no standard modal logic is it true that $\langle\varphi\rangle$ entails $\langle\Diamond\neg\varphi\rangle$ (special cases aside).

So the Possibility Principle is implausible on its face, and the argument Armstrong gave for it is unconvincing. I think we should conclude that the principle should be rejected.[14]

Happily, Armstrong also offered another and more attractive account of truthmaking and possibility. The idea is this. Suppose that we have (separately) two slices of bread, fifteen slices of cheese, and two slices of tomato. These things could have constituted a cheese and tomato sandwich—although in fact they don't. Plausibly, they together form a truthmaker for the proposition that a cheese and tomato sandwich *could* exist. Armstrong wrote:

> Consider, in particular, the cases where the entities in question do not exist, where they are *mere* possibilities. It is, let us suppose, true that ⟨it is possible that a unicorn exists⟩. What then is a minimal truthmaker for this truth? The obvious solution is combinatorial. The non-existent entity is some non-existent (but possible) combination out of elements that do exist. The phrase "non-existent combination" may raise eyebrows. Am I committing myself to a Meinongian view? No, I say. The *elements* of the combination are, I assert, the only truthmakers that are needed for the truth that this combination is possible. (T&T, 91–92)

I think that this is a very attractive account of what the truthmakers are for *some* truths about possibility (including truths about unicorns and tomato sandwiches).[15] However, it is doubtful that the combinatorial approach provides us with sufficient truthmakers for all the possibility claims made by the modal structuralist mathematician. The modal structuralist will assert that second-order ZFC could have had a model, but it seems unlikely that such a model could be created by recombining physical objects, because it seems unlikely that there are *enough* physical objects to go around. If, for example, there are only $\beth_3$ physical objects, we will not by combining them

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14 Armstrong (2007) recognized the error in his argument for the possibility principle and went on to offer a *new* argument for the Possibility Principle. For criticism of this later argument, see Pawl (2010).

15 For some criticisms of Armstrong's "combinatorialist" theory of modality, see Wang (2013).

be able to produce a set with $\beth_4$ elements, but *every* model of second-order ZFC contains sets with $\beth_4$ elements—and indeed much larger sets to boot. The problem of size has reemerged in a new form.

5 Armstrong on Truthmaking and Necessity (Part 1: Truth and Truthmakers)

Let's turn to Armstrong's discussion of propositions about necessity. Since our concern is Armstrong's philosophy of mathematics, we need not discuss all of Armstrong's views about truthmaking and necessity. Instead, we'll focus on what he had to say about truthmakers for the theorems of mathematics. Armstrong's discussions of this topic in *Truth and Truthmakers* and *Sketch for a Systematic Metaphysics* are very different. In this section, I'll consider chapter eight of *Truth and Truthmakers,* leaving the later book until section 7.

Armstrong (T&T, 99, 111) suggested that the numbers themselves constitute truthmakers for some arithmetical truths. For example, seven, five and twelve may together form a truthmaker for $\langle 7 + 5 = 12 \rangle$.[16] For Armstrong, so long as seven, five and twelve exist they *must* be related in this way, and so nothing *beyond* their existence is needed to explain their being so related. This relation between the three numbers is "internal" to them.

This is an important idea, and I will return to it in section 6. But this is not on its own a complete solution to the problem at hand. Consider for example the proposition $\langle \beth_\omega + \beth_\omega = \beth_\omega \rangle$. This is a theorem of orthodox mathematics, and so Armstrong would surely accept that the proposition is true, when given its proper modal interpretation. But what is its truthmaker? Surely $\beth_\omega$ itself can be a truthmaker only if it exists. However, as we saw in section 1 and section 2, it is doubtful for Armstrong that $\beth_\omega$ exists.[17]

Later in the chapter, Armstrong discussed analytic truths. He wrote:

> A traditional view, which has many supporters, is that [analytic] truths are true solely in virtue of the meanings of the terms in which they are expressed. (T&T, 109)

Armstrong went on to say that "[t]he phrase 'in virtue of' inevitably suggests truthmakers." So Armstrong proposed that if a sentence S is analytic, the

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16 A variant on this suggestion (T&T, 98) is that any truthmaker for $\langle 7$ exists$\rangle$, $\langle 5$ exists$\rangle$ and $\langle 12$ exists$\rangle$ will also be a truthmaker for $\langle 7 + 5 = 12 \rangle$.

17 Of course, for Armstrong, "$\beth_\omega$ exists" is true when given a suitable modal reinterpretation—but this is only because, so interpreted, the sentence doesn't actually assert the existence of $\beth_\omega$!

proposition it expresses is made true by the meanings of the words in S. For example, ⟨A father is a male parent⟩ is made true by the meanings of "a", "father", "is", "a", "male" and "parent".

Now Armstrong suggested—somewhat tentatively—that statements in mathematics about what is necessary are analytic.[18] On this view, the meanings of mathematical terms make true all such statements.

I do not dismiss completely the claim that mathematical truths are analytic.[19] However, Armstrong's version of this thesis is insufficient to solve the problem at hand. Let p be some true proposition from pure mathematics. Armstrong believed that the theorems of pure mathematics are *necessarily* true. So we can ask what the truthmaker for p *would* have been, had there been no language-users. How might Armstrong reply? Surely it is not adequate to say that the meanings of English words would have been the truthmakers—for English words would not have existed in the absence of English speakers.[20] If Armstrong replies that p would not have had a truthmaker, he would be stuck with the surely unwanted conclusion that it is possible for a proposition to be true without a truthmaker. And so the proper Armstrongian conclusion is that p would have had a different set of truthmakers, had there been no language-users. But then we are left with the question of what these truthmakers would have been — and until this question is answered, Armstrong's account is incomplete.

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18 Armstrong wrote: "There may be something mechanical, something purely conceptual, purely semantic, in the deductive following-out of proofs of the existence of the possible. (See the account of analytic truth to come in 8.9.)" (T&T, 102). Note that this quotation is from the chapter on *necessary* truths in T&T. So I take it that what Armstrong is (tentatively) suggesting here is that truths in mathematics about what is necessary are analytic.

19 When Armstrong says that "a traditional view" is that analytic truths are "true solely in virtue of the meanings of the terms in which they are expressed," his wording seems to derive from the introduction to Ayer (1946). I think that few philosophers of mathematics today would defend Ayer's view in all its details. However, there are still philosophers who endorse views which resemble Ayer's position in important respects. See for example Rayo (2013).

20 Perhaps some philosophers will insist that words (or their meanings) are necessarily existing abstract objects. However, I take it that *Armstrong* would not take this line. As we've seen, Armstrong rejected necessary abstracta.

6 Armstrong on Truthmaking and Necessity (Part 2: Sketch for a Systematic Metaphysics)

By the time he wrote *Sketch for a Systematic Metaphysics,* Armstrong had decided to reject his earlier suggestion that mathematical truths are analytic, saying that such a view implies that mathematics is "too arbitrary or conventional" (SSM, 91). But he suggested an alternative approach, which we will now consider.

We should begin by looking at Armstrong's metaphysics of law. Armstrong claimed that a law is a relation between properties (SSM, 35). Here is a toy example. Suppose that it is a law that being dehydrated causes headaches. For Armstrong, this means that a certain relation (viz. $\mathcal{N}$, the nomic relation) obtains between two properties (viz. the property *is dehydrated*, and the property *has a headache*). Armstrong would symbolize this as follows:[21]

$\mathcal{N}$ (*is dehydrated*, *has a headache*)

Now Armstrong claimed that laws have "instantiations". Our law, for example, is instantiated whenever someone is dehydrated and, consequently, has a headache. A law, on this view, is itself a property. And we have already seen that Armstrong was happy to posit "immanent" properties. This led Armstrong to the view that every law is instantiated. He wrote:

> If laws are a species of universal, then, according to me at least, they have to be instantiated at some place and time. Well, we talk of laws being instantiated, do we not? (The points where the laws are 'operative'.) So this instantiation of laws is the instantiation of a special sort of universal. (Note that this would require every law to be somewhere instantiated in space-time.) […] One consequence of this is that there cannot be laws that are never instantiated. (SSM, 41)

Now Armstrong suggested that it is certain mathematical and logical laws which make true the necessities of mathematics.

He began his discussion of this proposal by appealing to his Entailment Principle, arguing that truthmakers for the axioms of a mathematical theory

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21 I allow myself here to omit some of the finer details of Armstrong's account—in particular, I do not mention "state of affairs types" (SSM, 28–40).

must also be truthmakers for the theorems (SSM, 90). This manoeuvre is suspect. While it is known that the theorems of orthodox mathematics are entailed by the axioms (that's what makes them theorems, after all) it is far from clear that the axioms entail* the theorems.[22]

And it gets worse. To complete his account, Armstrong still needed to specify truthmakers for the axioms of our mathematical theories. To do this, he appealed to his theory of laws:

> We do, of course, have to recognize that introducing the Entailment Principle drives us back to consider the axioms from which mathematical systems are developed. [...] I suggest that we should postulate laws in logic and mathematics (non-contradiction, excluded middle in logic, Peano's axioms for number, or whatever laws logicians and mathematicians wish to postulate). In the light of the nature of proof just argued for we might suggest that such laws might be all we needed to postulate in the way of an ontology for logical and mathematical entities. (SSM, 90–91)

It is not credible, as Armstrong suggests here, that the Peano axioms are "laws" in Armstrong's sense. For example, one of the Peano axioms states that the natural numbers are *unending* in the sense that every natural number has a successor.[23] For Armstrong, this statement may not be true when taken at face value. For Armstrong, as we've seen, the existence of very large natural numbers is doubtful, and it is at least possible that there is a *largest* natural number, which has no successor. To circumvent this point, Armstrong will presumably insist on a modal reinterpretation of the axiom. On this view, the axiom, properly interpreted, states that, necessarily, every model of the Peano axioms is unending.

The corresponding Armstrongian law would then have to be:

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22   Suppose, for example, that Linnebo (2017) is correct and entailment* coincides with intuitionistic entailment. Then consider some statement φ which is provable classically but not intuitionistically from the prevailing axioms. (For example, φ might be $(\psi \lor \neg\psi)$, where ψ is some statement independent of the prevailing axioms.) Assuming, with Armstrong, that the inferences of classical logic are all truth-preserving, and the axioms of orthodox mathematics are true, we can conclude that φ is true. However, because it is not entailed* by the prevailing axioms, we cannot identify a truthmaker for it using the proposed method.

23   The "successor" of a natural number is the number that comes *immediately after it,* when the natural numbers are arranged in the customary fashion. So for example the successor of nineteen is twenty.

$\mathcal{N}$(*is a model of Peano arithmetic*, *is unending*)

But this is problematic, because Armstrong believed that all laws are instantiated in the physical world—and it is far from clear that this law is instantiated. It may be that the physical world is finite. However, every model of Peano arithmetic is infinite. And so it may be that there are no physical models of Peano arithmetic, in which case the above-mentioned law is uninstantiated.

We might be able to avoid this problem by arguing on empirical grounds that there are infinitely many physical objects. For example, we might appeal to the common (though admittedly contested) assumption in physics that space is infinitely divisible. However, the problem that I have just described will reassert itself when we turn our attention from Peano arithmetic to other branches of mathematics which posit a greater number of entities. The most extreme case is set theory. Any model of second-order ZFC would have to have a truly vast domain, containing $\beth_\omega$ elements and more. There is no empirical reason to think that there exist that many physical objects. So we are left with the conclusion that the Armstrongian laws corresponding to the axioms of set theory are uninstantiated.

To avoid these problems, an Armstrongian would have to list a number of basic principles for mathematics which express laws that *are* instantiated in the physical world, and argue that they entail* the truths of mathematics. I don't know that this impossible, but it is far from obvious that it can be done. And even if it *could* be done, the problem of identifying truthmakers for facts about what is possible would remain.

7 An Alternative Approach

Let's review. Armstrong believed that mathematical entities are located within the physical world. For example, wherever there is a pair of things, there is the number two. However, Armstrong realized that the physical world is not large enough to accommodate all the entities posited by modern pure mathematics. So he adopted a modal interpretation of mathematics. For Armstrong, pure mathematics tells us not about what is, but about could be and must be. However, Armstrong believed that every truth has a truthmaker within the physical world, and so he was left with the unenviable task of identifying truthmakers for modal truths within the physical world. I have

argued that he did not succeed.[24] In this final section, I would like to put forward an alternative approach—an approach which will, I hope, appeal to those impressed by Armstrong's metaphysical system.[25]

Armstrong accepted a version of the methodological principle known as Occam's razor. He rejected mathematical Platonism largely for this reason. A "Platonic realm of numbers," he wrote, is an "ontological extravagance" (T&T, 100).[26] However, Armstrong did not use his razor to excise supervenient entities. Supervenient entities, he thought, are an "ontological free lunch". For example, he did not think that universalism in mereology is objectionably unparsimonious:

> Whatever supervenes or, as we can also say, is entailed or necessitated, is not something ontologically additional to the subvenient, or necessitating, entity or entities. [...] The terminology of "nothing over and above" seems appropriate to the supervenient. [...] If the supervenient is not something ontologically additional, then this gives charter to, by exacting a low price for, an almost entirely permissive mereology. Do the number 42 and the Murrumbidgee River form a mereological whole? [...] The whole, if it exists, is certainly a strange and also an uninteresting object. But if it supervenes on its parts, and if as a consequence of supervening it is not something more than its parts, then there seems no objection to recognizing the whole. So in this essay permissive mereology, unrestricted mereological composition, is embraced. (1997, 12–13)

On an uncharitable interpretation of this passage, Armstrong's view was that if the existence of x necessitates the existence of y, then y is "nothing over and above" x. But this is hardly plausible. Perhaps God exists necessarily, but it would be grossly immodest for me to claim that God is nothing over and above me. Perhaps I could not have had different parents, in which case my existence necessitates theirs. But they would quite properly take exception to the suggestion that they are nothing ontologically additional to me.[27]

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24 It is worth noting in passing that Armstrong's theory of *propositions* was problematic in rather similar ways. On this point, see McDaniel (2005).

25 For a very different approach, see Read (2010).

26 Armstrong also had epistemological reasons for rejecting Platonism (SSM, 2). For lack of space, I do not discuss epistemology in this paper.

27 For a more detailed discussion of these points, see Schulte (2014).

Cameron (2008) has suggested a more promising way of developing Armstrong's idea that supervenient entities are "free". To put it briefly, Cameron's proposal is as follows. Compare the following two propositions:

⟨*m* exists.⟩ (where *m* is a marriage, between Ashni and Ben)

⟨*e* exists.⟩ (where *e* is an electron)

The former proposition is made true by certain patterns of human activity—involving perhaps Ashni, Ben, a registrar, some pieces of paper, and some metal rings. Ashni and Ben's marriage is a *derivative* entity: its existence is explained by facts about things other than itself. The electron *e* is not derivative. The electron's existence is not explained by facts about other things; *e* itself is the only truthmaker for the proposition ⟨*e* exists⟩.[28]

More generally, Cameron's proposal is this. When *x* is fundamental, the only truthmaker for ⟨*x* exists⟩ is *x* itself. When *x* is derivative, ⟨*x* exists⟩ has a truthmaker other than *x* itself.[29]

Cameron adds that it is derivative entities *in this sense* that are an "ontological free lunch," to use Armstrong's phrase. In effect, Cameron replaces the familiar slogan "Do not multiply entities beyond necessity" with a variant: "Do not multiply *fundamental* entities beyond necessity." Since mereological compounds are non-fundamental, Cameron infers, mereological universalism is not objectionable on grounds of parsimony.[30]

Cameron briefly suggests an application of this idea to impure set theory. He proposes that that an impure set is "nothing over and above" its elements, so there is no objection on grounds of parsimony to positing all those impure sets that can be built up from *basic elements* whose existence can already be established. On this view, there is no need to re-interpret set theory in a "possibilist" manner. We maintain that all the sets posited by set theorists really

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28 Sharp-eyed readers will note that in this section I assume an *explanatory* conception of truthmaking, according to which, when *T* is a truthmaker for *p*, *T explains the truth of p*. For discussion, see Cameron (2018).

29 I have actually modified Cameron's proposal in a small way. Cameron's view is that when *x* is derivative, *x* is *not* a truthmaker for ⟨*x* exists⟩. I find this claim puzzling (How could *x* fail to make true ⟨*x* exists⟩?) and since it is inessential to my argument, I omit it.

30 Suppose that *a* and *b* are fundamental objects, and that (*a* + *b*) is their mereological sum. According to Cameron *a* and *b* *collectively* make true ⟨(*a*+*b*) exists⟩. For Cameron, this proposition has no *single* truthmaker; rather there are some things which *together* make the proposition true. This is a subtlety of Cameron's view which I ignore in the main text, for simplicity.

do exist, although they are not fundamental. Let's develop this Cameronian proposal in more detail.

Why does the set {Jill, Joe} exist? I suggest that it exists because Jill exists, and because Joe exists—and that is all. Nothing more is needed. And so, I suggest, any truthmaker for ⟨Jill exists⟩ and ⟨Joe exists⟩ will also be a truthmaker for ⟨A exists⟩, where A is {Jill, Joe}. More generally:

1. If T is a truthmaker for ⟨x exists⟩, for each x in a non-empty set X, then T is a truthmaker for ⟨X exists⟩ also.[31]

So much for propositions about the existence of sets. But a complete truthmaker theoretic account of the sets will also include an account of what the truthmakers are for other propositions, including propositions about the identity and distinctness of sets, and propositions about what is an element of what.

Let's start with identity. Suppose that someone asks us why Joe is identical to Joe—that is, we are asked why Joe is identical with himself. This is a very peculiar question. The best answer to it that I can come up with goes like this. For Joe to bear the identity relation to himself, it suffices that he exists. Self-identicality is not some additional characteristic that requires further explanation. Joe exists, and so he is self-identical. And that is that. If this is right, I suggest, any truthmaker for the proposition ⟨Joe exists⟩ must also be a truthmaker for ⟨Joe = Joe⟩. In general, any truthmaker for ⟨x exists⟩ will also be a truthmaker for ⟨$x = x$⟩.[32]

Something similar is plausible in the case of non-identity. If, bizarrely, we are asked why it is that Jill is not identical with Joe — if we are asked why they are two people and not one—all we can say in reply is that to be non-identical

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31 What about the empty set? One might be tempted to avoid the problem by denying that the empty set exists. This proposal, as Hazen (1991) argues, is less radical than it might first seem, and Armstrong did in some places express scepticism about the empty set (T&T, 114). However, given Armstrong's usual hostility towards philosophically motivated reforms to standard mathematical practice, I think it desirable, from an Armstrongian point of view, to preserve the empty set. So here is an alternative, inspired by Kit Fine's well-known discussion of zero-grounding (2012).

   Armstrong generally supposed that a truthmaker will always be a *single thing*. But we might want to allow that a proposition can be made true by two things acting in concert, or three things, or four things, or more. For example, we might say that a, b and c collectively make true ⟨{a, b, c} exists⟩. Taking this line of thought still further, we could argue that in some unusual cases a proposition is made true by *zero* things; as we might put it, such propositions are *trivially* made true. On this view, we may say that ⟨∅ exists⟩ is trivially made true.

32 For more detailed discussion of the question of how truths of identity are to be explained, see A. Burgess (2012) and Shumener (2017).

the two Bidens need only exist. Jill exists. She is one person. Joe exists too. He is another. And that is all. There is nothing extra that Jill and Joe need to do or to be in order to be distinct—existing is enough. And so, I suggest, any truthmaker for ⟨Jill exists⟩ and ⟨Joe exists⟩ is a truthmaker also for ⟨Jill ≠ Joe⟩. More generally, if x and y exist and are distinct, any truthmaker for ⟨x exists⟩ and ⟨y exists⟩ must also be a truthmaker for ⟨$x ≠ y$⟩.

I want to recommend a similar treatment of the relations of membership and non-membership. If we are asked why Joe is an element of his singleton, there is nothing we can say except that, for this to be so, it suffices that Joe and his singleton exist. No more is needed. And if we are asked why Joe is not an element of {Jill}, we can say only that it is enough that Joe and {Jill} exist. More generally, I suggest, if x is an element of Y, then any truthmaker for ⟨x exists⟩ and ⟨Y exists⟩ is a truthmaker too for ⟨x ∈ Y⟩. And if x is not an element of Y, though they both exist, any truthmaker for ⟨x exists⟩ and ⟨Y exists⟩ is a truthmaker too for ⟨x ∉ Y⟩.

Let me put all of this in a rather different way. Let's say that a relation R is "strongly internal" if and only if the following condition is met: Necessarily, for any a and b, if a bears R to b then (1) a bears R to b at any world at which a and b both exist, and (2) at every such world, any truthmaker for ⟨a exists⟩ and ⟨b exists⟩ is also a truthmaker for ⟨a bears R to b⟩.[33] If R is a strongly internal relation and a bears R to b, then no explanation for this is required, beyond whatever is needed to account for the fact that the relata exist. My proposal is that the relations of identity, non-identity, membership and non-membership are strongly internal in this particular sense.[34] In summary:

1. If T is a truthmaker for ⟨x exists⟩, for each x in a non-empty set X, then T is a truthmaker for ⟨X exists⟩ also.

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33 Armstrong said that a relation is internal if "given just the terms of the relation, the relation between them is necessitated" (T&T, 9). That is, given any relation R, R is internal (in Armstrong's sense) just in case the following is necessary: For any a and b, if a bears R to b, then at every world at which a and b exist, a bears R to b.

   Clearly, any strongly internal relation is also internal in Armstrong's sense.

   The converse, however, is open to dispute. Suppose *arguendo* that God exists necessarily. Then the relation *x and y are such that God exists* is internal, in Armstrong's sense. But it is doubtful that this relation is strongly internal, for it is hardly plausible that any truthmaker for ⟨Joe exists⟩ and ⟨Jill exists⟩ mustalso be a truthmaker for ⟨God exists⟩.

34 Note that strongly internal relations need not be universals —they may be "second-rate" properties. In saying that non-membership is strongly internal, I do not assert that it is a genuine universal.

2. If a relation R is "strongly internal", then whenever a bears R to b, any truthmaker for $\langle a$ exists$\rangle$ and $\langle b$ exists$\rangle$ is also a truthmaker for $\langle a$ bears R to $b \rangle$.

3. The relations of identity, non-identity, membership and non-membership are strongly internal.

Let's take this further. Suppose for example that T is a truthmaker for the proposition $\langle$Jill exists$\rangle$. Then by (1), T is also a truthmaker for each of these propositions:

$\langle S_1$ exists$\rangle$, where $S_1 = \{$Jill$\}$

$\langle S_2$ exists$\rangle$, where $S_2 = \{\{$Jill$\}\}$

$\langle S_3$ exists$\rangle$, where $S_3 = \{\{\{$Jill$\}\}\}$

...

...

By (1) again, T is also a truthmaker for $\langle S_\omega$ exists$\rangle$, where S_ω is the set $\{$Jill, S_1, S_2, S_3, ... $\}$.

By (2) and (3), T is a truthmaker too for various propositions about the relations among these sets, propositions like $\langle$Jill $\neq S_1 \rangle$, $\langle S_1 = S_1 \rangle$, $\langle S_1 \neq S_\omega \rangle$, $\langle S_\omega = S_\omega \rangle$, $\langle$Jill $\in S_1 \rangle$, $\langle S_1 \in S_\omega \rangle$, and $\langle S_1 \notin$ Jill$\rangle$.

We can go further still, into the uncountable. Given our account, T will be a truthmaker for $\langle S^*$ exists$\rangle$, where S^* is the set of non-empty subsets of S_ω. S^* is an uncountable set. And T will be a truthmaker for $\langle S^{**}$ exists$\rangle$, where S^{**} is the set of non-empty subsets of S^*—a set even larger than S^*. And proceeding in this way, we can locate in the physical world truthmakers for propositions concerning sets at all levels of the vertiginous set-theoretic hierarchy, including sets of arbitrarily high cardinality.

And what of Armstrong's claim that all entities exist "somewhere, somewhen" (SSM, 15)? Well, some readers may find it edifying to insist that a set is located wherever its elements are.[35] On this view, you are co-located with your singleton, and its singleton, and *its* singleton, and so on *ad infinitum.* I

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[35] Maddy (1990) defends this view. I lack the space for a thorough treatment of Maddy's approach, but I would like note in passing that Maddy's version involves *reforms* to standard set theory: to be specific, Maddy identifies individuals with their singletons (e.g. for Maddy, Socrates = {Socrates})

offer no objection to this proposal. But I find it hard to see how to *justify* the claim that sets have spatial locations, and more importantly it seems to me that we need not endorse this claim to earn the title "naturalist." We insist that all fundamental objects are physical, and that all truths have physical truthmakers—and this is naturalism enough.

Back to the cardinal numbers. According to the current proposal, even if the fundamental objects are rather few, nevertheless the sets are fantastically numerous. This allows us to maintain Armstrong's original account of cardinal number without having to worry about the problem of size, and without recourse to possibilism. Given the current proposal, for example, $\beth_\omega$ *is* instantiated in the hierarchy of sets, even if there are only finitely many fundamental entities. If we add that it is not possible for there to be nothing,[36] we are left with the conclusion that the cardinal numbers exist necessarily.

Of course, a thorough truthmaker-theoretic account of mathematics would also cover functions, complex numbers, matrices, ordinal numbers, graphs, and all the other mathematical creatures. You will probably be relieved to hear that I don't intend to deal with all these topics now. It's time for a cup of tea, after all. But I hope that my discussion of sets and cardinals is sufficient to motivate cautious optimism about Armstrongian naturalism—despite the errors of detail that we have identified in Armstrong's discussions of the metaphysics of mathematics.[*]

Thomas M. E. Donaldson
Simon Fraser University
tmdonald@sfu.ca

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and she eschews pure sets. The Armstrongian approach that I recommend preserves set theory in its usual form. (On the issue of pure sets, see footnote 31).

36 Armstrong changed his mind on the question whether it is possible for there to be nothing. In Armstrong (1989, chap. 4, section IV) he claims that this is possible, but in T&T (105) he retracts the claim.

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